

Vortex structures of rotating Bose-Einstein condensates in anisotropic harmonic potential

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We found an analytical solution for the vortex structure in a rapidly rotating trapped Bose-Einstein condensate in the lowest Landau level approximation. This solution is exact in the limit of a large number of vortices and is obtained for the case of anisotropic harmonic potential. For the case of symmetric harmonic trap when the rotation frequency is equal to the trapping frequency, the solution coincides with the Abrikosov triangle vortex lattice in type-II superconductors. In a general case the coarse grained density is found to be close to the Thomas-Fermi profile, except the vicinity of edges of a condensate cloud.

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Bose-Einstein condensates (BEC) are a new state of matter where various aspects of macroscopic quantum physics can be studied. The experimental studying of BEC in ultra-cold rotating atomic gases shows a wide variety of new features in the physics of quantized vortices and vortex arrays [1–3] which can be not accessible in other quantum systems containing vortices, such as superfluid helium or type-II superconductors. In a harmonically trapped condensate rotating at a frequency close to the trap frequency, vortices form a triangular Abrikosov lattice and the coarse grained density approaches an inverted parabola [4–7]. At very fast rotation, when the number of vortices becomes close to the number of atoms, the states are strongly correlated and the vortex lattice is expected to melt.

We propose below the solution for the vortex structure of rotating BEC in anisotropic harmonic trapping. The found function satisfies the projected to the lowest Landau level (LLL) Gross-Pitaevskii (GP) equation with extremely high accuracy for a large number of vortices.

We consider a system of bosonic atoms strongly confined in the z direction by an external trapping potential such that the bosons become essentially two-dimensional. The bosons are confined in the plane by a anisotropic harmonic trapping potential $V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2)/2$, with $\omega_y < \omega_x$, for a definiteness, and the trap is rotating around the z axis with frequency Ω . We assume that all particles are in the same macroscopic quantum state described by the wave function $\psi(\mathbf{r})$. In the rotating frame the Gross-Pitaevskii equation for $\psi(\mathbf{r})$ reads:

$$\frac{\hat{\mathbf{p}}^2}{2m}\psi + g|\psi|^2\psi + V(\mathbf{r})\psi - \Omega\hat{L}_z\psi = \mu\psi, \quad (1)$$

where $\hat{\mathbf{p}}$ is the momentum operator, m is the particle mass, $g > 0$ is an effective 2D coupling constant, \hat{L}_z is the operator of the orbital angular momentum, μ is the chemical potential, and ψ is normalized to the total number of particles N .

As it is well known, the single-particle Hamiltonian H_0 for rotating neutral atoms is equivalent to the Hamiltonian of a charged particle in a uniform magnetic field $B = 2m\Omega$ along the z axis. The half the cyclotron frequency is equal to $\omega_c = \Omega$, and the vector-potential in the symmetric gauge is $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2 = m\Omega \times \mathbf{r}$.

$$\begin{aligned} H_0 &= \frac{\hat{\mathbf{p}}^2}{2m} - \Omega\hat{L}_z + V(\mathbf{r}) = \frac{1}{2m}(\hat{\mathbf{p}} - \mathbf{A}/c)^2 \\ &+ \frac{1}{2}m(\omega_x^2 - \Omega^2)x^2 + \frac{1}{2}m(\omega_y^2 - \Omega^2)y^2. \end{aligned} \quad (2)$$

Below the critical rotation frequency, $\Omega < \omega_x, \omega_y$, the presence of the residual confining potential lifts the degeneracy of the Landau levels. Provided that $\omega_x - \Omega, \omega_y - \Omega \ll \Omega$ and interactions are much smaller than the cyclotron frequency ($ng \ll 2\Omega$, where n is the two-dimensional particle density), we can restrict our consideration to the lowest Landau level.

The Gross-Pitaevskii equation projected onto the LLL of an asymmetric harmonic trap can be written in terms of dimensionless variables \tilde{x}, \tilde{y} as [7]

$$\begin{aligned} (\mu - \omega_t^+)f(\zeta) &= \frac{\omega_t^+ - \omega_t^-}{2}(-\sinh(2\nu)f''(\zeta) \\ &+ 2\zeta f'(\zeta)) + g\frac{1}{\pi}e^{-\tanh(\nu)\zeta^2/2} \\ &\times \int d\zeta' d\bar{\zeta}' e^{-2\zeta'\bar{\zeta}' + \zeta\bar{\zeta}' + \tanh(\nu)\zeta'^2 + \tanh(\nu)\bar{\zeta}'^2/2} f^2(\zeta')\overline{f(\bar{\zeta}')}. \end{aligned} \quad (3)$$

where

$$\zeta = \tilde{x} + i\tilde{y} = \frac{i}{2}\sqrt{\omega_t^+ - \omega_t^-}\sqrt{\sinh 2\nu}\left[z\rho - \frac{\bar{z}}{\rho}\right], \quad z = x + iy, \quad (4)$$

$$\tanh \nu = \frac{\omega_t^+}{\omega_t^-}\sqrt{\frac{\omega_t^- - \omega_c^2}{\omega_t^+ - \omega_c^2}}, \quad \rho^2 = \sqrt{\frac{(\omega_t^- + \omega_c)(\omega_t^+ + \omega_c)}{(\omega_t^- - \omega_c)(\omega_t^+ - \omega_c)}}, \quad (5)$$

with $\tilde{\omega}_x^2 = \omega_x^2 - \omega_c^2$, $\tilde{\omega}_y^2 = \omega_y^2 - \omega_c^2$, $\omega_t^\pm = \sqrt{\omega_c^2 + (\frac{\tilde{\omega}_x \pm \tilde{\omega}_y}{2})^2}$, and $d\tilde{x}d\tilde{y} = (\omega_t^+\omega_t^-/\omega_c)dxdy$. The LLL wave functions have a form

$$\Psi(x, y) = f(\zeta)e^{-\frac{1}{2}\omega_t^+ z\bar{z}}e^{-\frac{1}{2}(az^2 + b\bar{z}^2)}, \quad (6)$$

with an analytic function $f(\zeta)$, and $a = \frac{1}{2}\rho^2(\omega_t^- - \omega_c)$, $b = \frac{1}{2}(\omega_t^- + \omega_c)/\rho^2$.

The anisotropy parameter $\tanh \nu$ is equal to zero for the cylinder symmetric trap: $\omega_x = \omega_y \geq \Omega$. The opposite quasi-one-dimensional case with a narrow channel geometry is achieved at $\tanh \nu \rightarrow 1$, when the rotation frequency becomes equal to the lowest trapping frequency: $\Omega \rightarrow \omega_y < \omega_x$. Both limited cases were considered in details in [7].

Note that for the case of $\omega_x = \omega_y = \Omega$ the free energy $\int dx dy [\psi^* H_0 \psi + g|\psi|^4/2 - \mu|\psi|^2]$ has the same form as the Ginzburg-Landau functional of a superconductor in the magnetic field near a phase transition. The important difference and simplification in our case is the constant "magnetic field" $2m\Omega$ (infinite penetration depth). Therefore, instead of three equations obtained by variation over ψ , B , plus the boundary condition, we have one equation with the normalization condition $\int dx dy |\psi|^2 = N$. Since the penetration depth of the "magnetic field" is infinity, properties of our system will be similar to properties of type-II superconductors. Therefore it is reasonable to expect that superconductors in the magnetic field and rotating BEC will have similar structure of vortex lattice. Indeed, we can see that the well known approximate solution for Abrikosov vortex lattice [8] was built by use of the lowest Landau level (LLL) wave functions, as well as the solution for the BEC. Moreover, an approximate solution for Abrikosov lattice in type-II superconductor becomes the exact solution for the considered model of rotating BEC in the LLL approximation.

For the general case of asymmetric harmonic potential the ground state contains an ordered vortex structure in the parameters region $Ng/l^2/(\hbar(\omega_{x,y} - \Omega)) \gg 1$, where $l = (\hbar/m\Omega)^{1/2}$ is the effective magnetic length. The number of vortices will increase with the increase of this ratio. When the number of particles will become of the order of the number of vortices, the vortex lattice will melt through the phase transition to a strongly correlated state. In this region the Gross-Pitaevskii mean field equation is not applicable.

We will find the solution for the vortex structure as a special deformation of the exact solution for spatially homogeneous vortex lattice for the case of a cylinder symmetric potential ($\omega_x = \omega_y = \omega$) at the critical value of the frequency rotation $\Omega = \omega$:

$$f_0(\zeta) = \frac{(2v)^{1/4}}{\sqrt{S}} \vartheta_1(\pi\zeta/b_1, \tau) \exp(\pi\zeta^2/2v_c), \quad (7)$$

where S is the surface area, and ϑ_1 is the Jacobi theta-function given by

$$\vartheta_1(\eta, \tau) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n \exp\{i\pi\tau(n+1/2)^2 + 2i\eta(n+1/2)\}. \quad (8)$$

Real (u) and imaginary (iv) parts of the quasiperiod $\tau = u + iv$ will be fixed below. The Jacobi theta-functions are analytic in the complex plane and have zeros at the points $\eta = n\pi + m\pi\tau$, where n, m are integers. The function $f_0(\zeta)$ has zeros at the lattice sites $nb_1 + mb_2$, with $b_2 = b_1\tau$. These points correspond to the vortex locations. The parameter b_1 can be chosen real so that the area of the unit cell is $v_c = b_1^2 v$. Using the property of the ϑ -function:

$$|\vartheta_1(\pi\zeta/b_1, \tau)| = G(\tilde{x}, \tilde{y}) \exp\left[\frac{\pi\tilde{y}^2}{v_c}\right],$$

with a periodic oscillating G : $0 \leq G(\tilde{x}, \tilde{y}) \leq \vartheta_2(0, \tau)$, one can obtain that the envelope $|\overline{f_0(\zeta)}|$ has a polar symmetric form

$$|\overline{f_0(\zeta)}| \sim \exp\left[\frac{\pi(\tilde{x}^2 + \tilde{y}^2)}{2v_c}\right]. \quad (9)$$

The function $f_0(z)$ with fixed elementary cell area $v_c = \pi$ is an exact solution of Eq. (3) for the case of a critical rotation $\omega_x = \omega_y = \Omega$. The normalization coefficient in Eq. (7) is chosen such that the function $\Psi = [f_0(z)/l] \exp(-z\bar{z}/2)$ is normalized to unity. The function Ψ has a constant envelope and describes a periodic vortex structure. The minimum energy is obtained for the triangular lattice, where $\tau = \exp 2\pi i/3$, $v = \sqrt{3}/2$, and $b_1^2 = 2\pi/\sqrt{3}$. The chemical potential is then given by $\mu = \frac{\alpha Ng}{l^2}$, with $\alpha = (3^{1/4}/2) \sum_{b,c} (-1)^{mp} \exp\{-\pi^2(b^2 + c^2)/4b_1^2\} = 1.1596$, and $b = 2m$, $c = 2p$ being even integers.

In the general case ($\Omega < \omega_x, \omega_y$), we will find a solution of equation (3) in the form

$$f(\zeta) = (2v)^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n \hat{g}(a) \tilde{q}^{a^2} e^{i\sqrt{\pi v} a \zeta + \frac{\zeta^2}{2}(1-\tanh \nu)}, \quad (10)$$

where $a = 2n + 1$ are odd integers, $\tilde{q} = \exp[i\pi\tau/4]$, and $\hat{g}(a)$ is a differential operator acting on a . Substituting the trial function (10) into equation (3) for a triangular-like lattice (the lattice that becomes exactly triangular for $\omega = \Omega$)

we obtain:

$$\begin{aligned} & \left\{ \left(\mu - \omega_t^+ + (\omega_t^+ - \omega_t^-) \left[\frac{\tanh(\nu)}{1 + \tanh(\nu)} - \hat{A}\hat{A}^+ \right] \right) \hat{g}(a) \right. \\ & \quad \left. - \sqrt{v}g \sum_{b,c} \hat{g}(a-b)\hat{g}(a-c)\overline{\hat{g}(a-b-c)} \right. \\ & \quad \left. \times \exp \left[-\frac{\pi v}{4}(b^2 + c^2) \right] (-1)^{mp} \right\} \tilde{q}^{a^2} \exp[i\sqrt{\pi v}a\zeta] = 0. \end{aligned} \quad (11)$$

Here we introduced the operators \hat{A}, \hat{A}^\dagger which are creation (annihilation) operators for a corresponding harmonic oscillator with usual commutation relations $[\hat{A}, \hat{A}^\dagger] = 1$:

$$\hat{A}, \hat{A}^\dagger = \frac{\sqrt{\pi\tilde{v}}}{2}a \pm \frac{1}{\sqrt{\pi\tilde{v}}} \frac{\partial}{\partial a},$$

where $\tilde{v} = v\gamma$, with $\gamma = (1 + \tanh \nu)/(1 - \tanh \nu)$.

For large $\beta \equiv Ng/(l^2(\omega_t^+ - \omega_t^-)) \gg 1$ and $\mu^* \equiv (\mu - \omega_t^+)/(\omega_t^+ - \omega_t^-) - 1/(1 + \tanh \nu) \gg 1$ an approximate solution for $\hat{g}(a)$, which describes the vortex structure with a high accuracy, has the form

$$\hat{g}(a) = \frac{1}{\sqrt{\alpha\beta}} \sqrt{R^2 - \hat{A}^\dagger \hat{A}} \Theta(R^2 - \hat{A}^\dagger \hat{A}), \quad (12)$$

where Θ is the Heaviside step function, $R = \sqrt{\mu^*}$, and $\alpha = 1.1596$. We will see below that for the symmetric case R is the radius of the condensate cloud in units of l . Equation (12) is obtained taking into account that the leading contribution to the sum over b and c in Eq. (11) comes from small values of m and p , since already the contributions of terms with $|m| \geq 2$ or $|p| \geq 2$ are exponentially small. Provided that the dependence $\hat{g}(a)$ is smooth, which is the case for large R , we may consider large a and omit b and c in the arguments of the \hat{g} operators in Eq. (11). This immediately gives Eq. (12). Mathematically, the high accuracy of the solution is based on a known fast convergence of the series for ϑ -function due to the multiplier $\tilde{q}^{(2n+1)^2}$.

Substituting the solution (12) into Eq.(10) and expanding (10) into series in known eigenfunctions of harmonic oscillator we obtain after some algebra:

$$\begin{aligned} f(\zeta) &= \frac{(2v)^{1/4} \sqrt{(1 + \tanh \nu)}}{\sqrt{\alpha\beta}} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{[R^2]} (-1)^{[n(n-1)/2]} \\ &\times \sqrt{R^2 - k} \frac{(i\sqrt{\tanh \nu})^k}{2^k k!} \\ &\times H_k \left(\frac{\zeta}{\sqrt{\sinh 2\nu}} \right) H_k \left(\sqrt{\frac{\pi\tilde{v}}{2}} a \right) e^{-\pi\tilde{v}a^2/4}, \end{aligned} \quad (13)$$

where $H_k(w)$ are Hermite polynomials, $a = 2n + 1$. The solution (13) is simplified in the symmetric case, where $\nu \rightarrow 0$, $H_k(\zeta/\sqrt{\sinh 2\nu}) \rightarrow 2^{k/2} \zeta^k / \nu^{k/2}$, and in the one-dimensional case where $\nu \rightarrow \infty$ and operators A, A^\dagger becomes numbers: $A = A^\dagger \propto a$.

From the condition that the function $[f(\zeta)/l] \exp(-\zeta\bar{\zeta})$ is normalized to unity we find a relation $R = (2\alpha\beta\gamma/\pi)^{1/4}$. For the symmetric potential ($\gamma = 1$) this result is in agreement with Refs. [5, 6, 9–11]. As mentioned above the solution (13) includes limiting cases of cylinder and narrow channel geometries, considered in details in [7]. Numerical results [7] for these cases demonstrated excellent coincidence with the analytical solution. The structure of the vortex lattice for $R = 7$, $\tanh \nu = 1/4$ is shown in Fig. 1. Averaging the density over the oscillations, that is averaging the

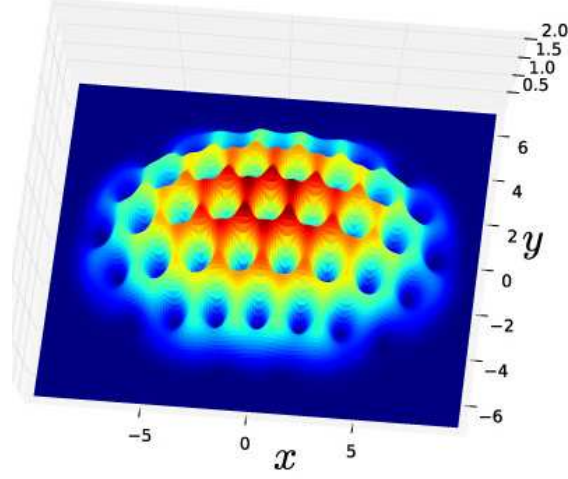


FIG. 1. (Color online) Condensate wave-function $|\psi(x, y)|^2$ for $R = 7$, $\tanh \nu = 1/4$. Coordinates x and y are given in units of l .

density $|\Psi|^2$ over a distance scale much larger than l , gives the coarse grained density:

$$\begin{aligned}
 \bar{n}_{cg}(r) &= \frac{n_{cg}(0)}{\cosh \nu} \sum_{k=0}^{[R^2]} \left(1 - \frac{k}{R^2}\right) \frac{(\tanh \nu)^k}{2^k k!} H_k \left(\frac{\zeta}{\sqrt{\sinh 2\nu}} \right) \\
 &\times H_k \left(\frac{\bar{\zeta}}{\sqrt{\sinh 2\nu}} \right) e^{-|\zeta|^2 + (\zeta^2 + \bar{\zeta}^2) \tanh(\nu)/2} \\
 &= \frac{n_{cg}(0)}{\pi \cosh \nu} \int \int_{-\infty}^{\infty} dt_1 dt_2 e^{-t_1^2 - t_2^2} \sum_{k=0}^{[R^2]} \left(1 - \frac{k}{R^2}\right) \frac{X^k}{k!} \\
 &\times e^{-|\zeta|^2 + (\zeta^2 + \bar{\zeta}^2) \tanh(\nu)/2} = \frac{n_{cg}(0)}{\pi \cosh \nu} \int \int_{-\infty}^{\infty} dt_1 dt_2 e^{-t_1^2 - t_2^2 + X} \\
 &\times \frac{\Gamma(1 + R^2, X) - X\Gamma(R^2, X)}{\Gamma(R^2 + 1)} e^{-|\zeta|^2 + (\zeta^2 + \bar{\zeta}^2) \tanh(\nu)/2},
 \end{aligned} \tag{14}$$

where $X(t_1, t_2) = 2 \tanh(\nu) (-i\zeta/\sqrt{\sinh(2\nu)} + t_1)(i\bar{\zeta}/\sqrt{\sinh(2\nu)} + t_2)$. In the limit $\nu \rightarrow 0$ when $X \rightarrow |\zeta|^2$, we obtain exactly the expression for the symmetric case [7]. The solution for a narrow channel case is obtained in the limit $\nu \rightarrow \infty$ taking into account the divergence of R : $R \propto (1 - \tanh \nu)^{-1/2}$.

The expression (14) can be simplified in the case of large R . Inside the condensate cloud we obtain the density profile which is close to the Tomas-Fermi inverted paraboloid:

$$\begin{aligned}
 n &= n_{cg}(0) \left[1 - \frac{1}{R^2(1 - \tanh^2 \nu)} [\tanh^2 \nu \right. \\
 &\quad \left. + \tilde{x}^2(1 - \tanh \nu)^2 + \tilde{y}^2(1 + \tanh \nu)^2] \right],
 \end{aligned} \tag{15}$$

with $n_{cg}(0) \approx 2N/(\pi R^2)$. The condensate is localized at the region inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ with semiaxes $a \geq b$:

$$a \approx R\sqrt{\gamma}, \quad b \approx \frac{R}{\sqrt{\gamma}}.$$

Outside the ellipse the density is negligible small and has the asymptotic form

$$\begin{aligned}
 n_{cg} &\propto \exp[R^2 \ln(|\zeta|^2(1 - \tanh \nu)^2) \\
 &\quad - \tilde{x}^2(1 - \tanh \nu) - \tilde{y}^2(1 + \tanh \nu)].
 \end{aligned} \tag{16}$$

Outside the condensate cloud the density drops exponentially at distance of the order of magnetic length or intervortex spacing. In the direction y , for example, in the interval $(1 \ll y - b \ll b^{1/3})$ we obtain

$$n_{cg} \sim n_{cg}(0) \frac{e^{-2(\bar{y}-b)^2}}{4\sqrt{2\pi}b(y-b)^2}. \quad (17)$$

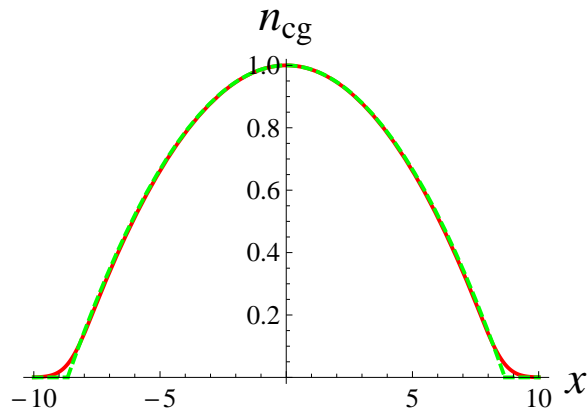


FIG. 2. (Color online) The coarse grained density $n_{cg}(x, 0)$ in units of \bar{n}_{2D} versus x , for $R = 5$, $\tan \nu = 1/2$. The dashed curve shows the Thomas-Fermi inverted-parabola shape, and x is given in units of l .

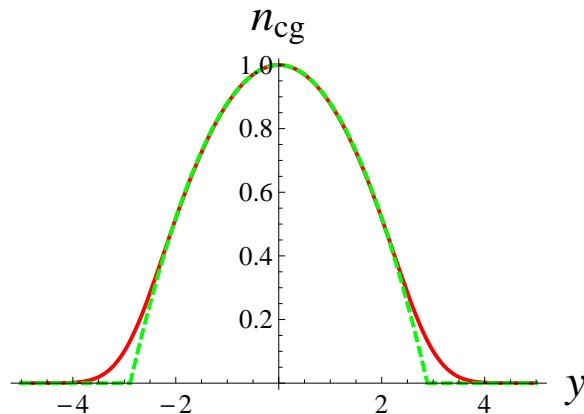


FIG. 3. (Color online) The coarse grained density $n_{cg}(0, y)$ in units of \bar{n}_{2D} versus y , for $R = 5$, $\tan \nu = 1/2$. The dashed curve shows the Thomas-Fermi inverted-parabola shape, and y is given in units of l .

At the Thomas-Fermi border we have $n_{cg} \sim n_{cg}(0)/\sqrt{2\pi|\zeta|^2}$. Our results show that inside the condensate cloud the density profile has the Thomas-Fermi inverted-paraboloid shape, except the region close to the edge: $|\delta z| \lesssim l$, where $|\delta z|$ is the distance to the border. We see in Figs. 2, 3 that for $R = 5$, $\tan \nu = 1/2$ the Thomas-Fermi formula works well already for $|\delta \zeta| \gtrsim 1$. The parabolic coarse grained density profile for the asymmetric potential was considered analytically and numerically in [12–14]. Deviations from the Thomas-Fermi density profile of $\bar{n}_{cg}(r)$ have been studied in Ref. [5] for the symmetric potential by using the variational numeric methods. Here we presented an analytic solution for the vortex lattice in a rapidly rotating BEC in an asymmetric harmonic trap. The found coarse grained density profile is close to inverted paraboloid form. The solution is asymptotically exact in the limit of a large number of vertices.

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